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Absence of higher order corrections to non-commutative Chern-Simons coupling

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Abstract: We analyze the structure of non-commutative pure Chern-Simons theory systematically in the axial gauge. We show that there is no IR/UV mixing in this theory in this gauge. In fact, we show, using the usual BRST identities as well as the identities following from vector supersymmetry, that this is a free theory. As a result, the tree level Chern-Simons coefficient is not renormalized. It also holds that the Chern-Simons coefficient is not modified at finite temperature. As a byproduct of our analysis, we prove that the ghosts completely decouple in the axial gauge in a non-commutative gauge theory.

Keywords: Chern-Simons Theories, BRST Symmetry, Non-Commutative Geometry.

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Contents

1. Introduction

The physics on non-commutative space (and space-time), after a new motivation from string theory, has attracted a lot of interest [1]. In addition to the string theory interests, non-commutative quantum mechanics and non-commutative field theories have their own attractions [2, 3] as well. The non-commutative space can be defined by

$$[\hat{x}^\mu, \hat{x}^\nu] = i\theta^{\mu\nu}\,, \tag{1.1}$$

where $\hat{x}^\mu$ are the space-time coordinate operators and $\theta^{\mu\nu}$ is a constant anti-symmetric tensor. In order to formulate non-commutative field theories, we need a realization of eq. (1.1) in the space of the fields (functions). In fact such a realization is simple to find, recalling the Weyl-Moyal correspondence [2], according to which the non-commutative version of a given field theory is obtained by replacing the usual product of functions by the $\star$-product:

$$f(x) \star g(x) = f(x)\, \exp\left(\frac{i}{2}\theta^{\mu\nu}\overleftarrow{\partial_\mu}\,\overrightarrow{\partial_\nu}\right) g(x)\,. \tag{1.2}$$

It has been discussed that the non-commutative space appears naturally in planar physics, namely, in $(2+1)$-dimensional theories. More specifically, the non-commutative plane accommodates the physics of charged particles in a (strong) magnetic field and, therefore, the quantum Hall physics [4, 5]. In particular, it has been shown that the non-commutative version of pure Chern-Simons theory captures the physics of Laughlin wave functions [4], in the sense that the level of the non-commutative U(1) Chern-Simons theory plays the role of the filling fraction. Being motivated by quantum Hall physics, non-commutative Chern-Simons theories, both abelian and non-abelian, have been studied extensively [6]–[10].

The perturbative analysis and the one-loop calculations of the non-commutative Chern-Simons theories have, thus far, been performed in covariant gauges [6, 7]. In the present work, following the lines of [11], we will study the non-commutative pure Chern-Simons in the axial gauge [12] and present explicit calculations at one loop. In the axial gauge, we have the advantage of ghost decoupling as well as various cancellations. In this way, we show, using the BRST as well as the vector supersymmetry of non-commutative Chern-Simons theories, that the one-loop result can be extended to all orders.

The paper is organized as follows. In section 2, we introduce the non-commutative Chern-Simons theory by giving its action and study the theory in the homogeneous axial gauge. Then, we show the BRST invariance and the vector supersymmetry of the gauge-fixed action. Using these symmetries we find the proper set of Ward identities, which allows us to conclude that this is a free theory. Explicit one-loop calculations are presented in section 3 where we also show that ghosts decouple in the axial gauge in any non-commutative gauge theory. The last section is devoted to some concluding remarks.

2. Non-commutative Chern-Simons theory

In this section, we discuss various aspects of the non-commutative pure Chern-Simons theory defined by

$$S_{\mathrm{NCCS}} = m \int_M \epsilon^{\mu\nu\lambda}\ \mathrm{Tr}\left(A_\mu \star \partial_\nu A_\lambda + \frac{2ig}{3} A_\mu \star A_\nu \star A_\lambda \right). \tag{2.1}$$

Here, the star product is defined in eq. (1.2), A_μ takes values in the Lie algebra $\mathrm{u}(N)$ (hermitean $N \times N$ matrices) and "Tr" stands for the trace over the $\mathrm{u}(N)$ indices in the representation where the generators satisfy $\mathrm{Tr}\ t^A t^B = \frac{1}{2}\delta^{AB}$. We assume that the base manifold M can be separated into a two-dimensional (non-commutative) subspace times R or S^1 and further assume that the $\star$-product (eq. (1.2)) can be defined on the two-dimensional part of this manifold. The $\star$-product with constant θ can only be realized on flat manifolds, namely, non-commutative plane, non-commutative cylinder and the non-commutative torus. Here we present our calculations for the case of a non-commutative plane, although the same arguments and results hold true also for non-commutative cylinder and torus.

The action in eq. (2.1) is invariant under the infinitesimal gauge transformations

$$A_\mu \longrightarrow A_\mu + \partial_\mu \lambda + ig\,(\lambda \star A_\mu - A_\mu \star \lambda) = D_\mu \lambda\,, \qquad \lambda \in \mathrm{u}(N)\,. \tag{2.2}$$

Under a finite gauge transformation, however, this action changes by a constant (the lagrangian density changes by a total derivative) and the consistency of the theory then requires that the coefficient of the Chern-Simons action should be quantized. We

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note that this result does not depend on θ and is also true for NCU(1) case. In fact, the $\theta \to 0$ limit is not a smooth one. This non-smoothness can be understood noting eq. (1.1) and the fact that, in the non-commutative case (no matter how small θ is), we deal with functions with countable number of degrees of freedom (for a proper representation, see e.g. [13]), while in the commutative case this is uncountable.

The pure Chern-Simons theory is a topological theory (both in the commutative and the non-commutative cases) since the action does not depend on a metric. However, a gauge fixing and the corresponding ghost actions would depend on a metric and one might wonder whether the topological nature of such theories will be destroyed by quantum corrections. In the case of a commutative pure Chern-Simons theory, it is known that the full theory including gauge fixing and ghosts is completely free (when suitably regularized) [11]. In fact, this is why the theory is topological. It is, therefore, interesting to ask if a similar conclusion holds for the non-commutative pure Chern-Simons theory. To analyze this question, let us recall that it is in the axial gauge that the free nature of the commutative pure Chern-Simons theory is best seen. In the non-commutative case, on the other hand, there is a possibility of UV/IR mixing [14] and it is well known that the axial gauge is not particularly suitable in the presence of IR divergences. Consequently, this question becomes even more interesting and non trivial in the present case.

The theory is defined in the homogeneous axial gauge $n^\mu A_\mu = 0$, where n^μ is a fixed direction in space, by adding a gauge fixing and ghost action of the forms (the definition of the covariant derivative is given in eq. (2.2) and, for simplicity, we will assume $n^2 = 1$)

$$S_{\rm gf} + S_{\rm ghost} = -\int_M \mathrm{Tr}\,\left(\frac{1}{\xi}(n\cdot A)\star(n\cdot A) + \bar{c}\star(n\cdot Dc)\right) \tag{2.3}$$

to our theory and looking at the theory in the limit $\xi \to 0$. Alternately, one can define the gauge fixing and the ghost actions with an auxiliary field (lagrange multiplier field) of the form

$$S_{\rm gf} + S_{\rm ghost} = -\int_M \mathrm{Tr}\,\left(F\star(n\cdot A) + \bar{c}\star(n\cdot Dc)\right). \tag{2.4}$$

The Feynman rules can be read out from the theory

$$S = S_{\rm NCCS} + S_{\rm gf} + S_{\rm ghost}\,. \tag{2.5}$$

We note that the gauge and the ghost propagators take the simple forms

$$\begin{aligned} D^{AB}_{\mu\nu}(p) &= \frac{i}{m}\,\delta^{AB}\epsilon_{\mu\nu\lambda}\,\frac{n^\lambda}{(n\cdot p)}\,, \\ D^{AB}(p) &= \frac{i\delta^{AB}}{(n\cdot p)}\,. \end{aligned} \tag{2.6}$$

Here $A, B = 0, 1, \ldots, N^2-1$ are the u(N) indices, which is the same as the commutative case (for more details, see [15]). It is important to note that, in the homogeneous axial gauge, the gauge propagator is transverse to n^μ, namely,

$$n^\mu D_{\mu\nu}^{AB}(p) = 0\,. \tag{2.7}$$

The interaction vertices can also be read out quite easily and have the forms

$$\begin{aligned}\text{Three gluon vertex: } \Gamma_{\mu\nu\lambda}^{ABC}(p_1,p_2,p_3) &= mg\epsilon_{\mu\nu\lambda}\, Q_\theta^{ABC}(p_1,p_2,p_3)\delta^3(p_1+p_2+p_3)\,,\\ \text{Ghost-gluon vertex: } \Gamma_{\mu}^{ABC}(p_1,p_2,p_3) &= gn^\mu\, Q_\theta^{ABC}(p_1,p_2,p_3)\delta^3(p_1+p_2+p_3)\,,\end{aligned} \tag{2.8}$$

where

$$Q_\theta^{ABC}(p_1,p_2,p_3) = \left(f^{ABC}\cos\frac{p_1\theta p_2}{2} - d^{ABC}\sin\frac{p_1\theta p_2}{2}\right). \tag{2.9}$$

For f^{ABC} and d^{ABC} factors and their normalizations we use the conventions of [15], i.e. if we denote the generators on u(N) by t^A, then,

$$t^A t^B = \frac{i}{2}f^{ABC}t^C + \frac{1}{2}d^{ABC}t^C\,, \tag{2.10}$$

where f^{ABC} is totally anti-symmetry and d^{ABC} completely symmetric. Also, $t^0 = \frac{1}{\sqrt{2N}}\mathbf{1}$ and $f^{0AB} = 0$ and $d^{0AB} = \sqrt{\frac{2}{N}}\delta^{AB}$.

It is immediately clear from the Feynman rules of the theory that because the ghost interaction vertex involves a factor of n^μ, and that the gauge propagator is transverse to this direction, there can be no quantum correction to the ghost two point and three point functions. In other words, the wave function and the vertex renormalization for the ghosts is trivial, $\tilde{Z}_1 = 1 = \tilde{Z}_3$. This also happens in the commutative gauge theories in the axial gauge. In fact, in the commutative theories, one can show that the ghosts decouple completely from the theory in the axial gauge. Here, we see that the ghosts decouple, at least in diagrams involving open ghost lines. It is not *a priori* clear that the ghosts would decouple in diagrams involving closed loops. We will come to this issue shortly.

2.1 BRST symmetry

Even though gauge fixing and ghost lagrangians break the u(N) gauge invariance of eq. (2.2), the full theory is invariant under the fermionic BRST symmetry

$$\begin{aligned}\delta A_\mu &= \omega(D_\mu c)\,,\\ \delta c^A &= ig\omega c \star c\,,\\ \delta\overline{c} &= -\omega F\,,\\ \delta F &= 0\,.\end{aligned} \tag{2.11}$$

Here, ω is a constant anti-commuting parameter of transformation and the invariance of the total action is easily seen in the formulation with the auxiliary field. It is in this formulation also that the BRST symmetry is nilpotent off-shell. One can now derive the Ward identities (Slavnov-Taylor identities) of the theory following from the BRST invariance in a standard manner. Without going into details, we simply note that the Ward identities take the form

$$\int_M \left(\frac{\delta\Gamma}{\delta A_\mu}\star\frac{\delta\Gamma}{\delta K^\mu}-\frac{\delta\Gamma}{\delta c}\star\frac{\delta\Gamma}{\delta K}+F\star\frac{\delta\Gamma}{\delta\overline{c}}\right)=0\,, \tag{2.12}$$

where K^μ and K are sources for the composite BRST variations $D_\mu c$ and $ic\star c$, respectively. This form of the Slavnov-Taylor identity holds in any gauge. However, it leads to very interesting consequences in the axial gauge. For example, as we have already noted, in the axial gauge ghosts decouple, at least in diagrams involving open ghost lines because of the structure of the ghost interaction as well as eq. (2.7). As a result, the terms in the effective action involving the sources K^μ and K do not renormalize either, much like the ghost fields and the ghost interaction vertex. An immediate consequence of this is the fact that, in the axial gauge, the wave function and the vertex renormalization for the gauge fields are equal, namely, $Z_1 = Z_3$. This does not, however, imply that they are trivial.

It is worth pointing out here that, in the axial gauge, the complete two point function can be parameterized, in general, as

$$\begin{aligned}\Gamma^{AB}_{\mu\nu}(p) = \delta^{AB}\Bigg(im\epsilon_{\mu\nu\lambda}p^\lambda(1+\Pi_2(p^2))+\left(p^\mu-\frac{p^2n^\mu}{n\cdot p}\right)\left(p^\nu-\frac{p^2n^\nu}{n\cdot p}\right)\times\\ \times\,\Pi_3(p^2)\Bigg)\,,\end{aligned} \tag{2.13}$$

where Π_2 and Π_3 represent quantum corrections to the two tensor structures. This implies that we can determine the Chern-Simons coefficient (including the quantum corrections) from the complete two point function as

$$m(1+\Pi_2(0))\,\delta^{AB} = -\frac{i}{6}\epsilon^{\mu\nu\lambda}\left.\frac{\partial\Gamma^{AB}_{\mu\nu}(p)}{\partial p^\lambda}\right|_{p=0}\,. \tag{2.14}$$

2.2 Vector supersymmetry

In addition to the BRST invariance, the total action possesses another fermionic symmetry, namely, under

$$\begin{aligned}\delta A_\mu &= \epsilon_{\mu\nu\lambda}\epsilon^\nu n^\lambda c\,,\\ \delta c &= 0\,,\\ \delta\overline{c} &= -\epsilon^\mu A_\mu\,,\\ \delta F &= \epsilon^\mu\partial_\mu c\end{aligned} \tag{2.15}$$

the total action can be easily checked to be invariant. Here, ϵ^μ is a constant anti-commuting vector parameter. It is interesting to note that, unlike the BRST symmetry, these symmetry transformations are linear in the field variables. Consequently, the corresponding Ward identities can be derived without the need for any extra

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sources (there are no composite variations) and take the form

$$\int_M \left(\epsilon^{\mu\nu\lambda}\frac{\delta\Gamma}{\delta A^\nu}\star c\, n^\lambda - \frac{\delta\Gamma}{\delta F}\star\partial^\mu c + A^\mu\star\frac{\delta\Gamma}{\delta\overline{c}}\right) = 0\,. \tag{2.16}$$

While the usual Slavnov-Taylor identities in eq. (2.12) only imply $Z_1 = Z_3$ (and nothing about the triviality of these quantities) in the axial gauge, the identities following from the vector supersymmetry lead to the fact that this is a free theory. For example, differentiating eq. (2.16) with respect to $\frac{\delta^2}{\delta A^\mu \delta c}$ and setting all fields equal to zero, we obtain (in momentum space),

$$\epsilon^{\mu\nu\lambda}\,\frac{\delta^2\Gamma}{\delta A^\mu(-p)\delta A^\nu(p)}\,n_\lambda + ip^\mu\,\frac{\delta^2\Gamma}{\delta A^\mu(-p)\delta F(p)} - 3\,\frac{\delta^2\Gamma}{\delta\overline{c}(-p)\delta c(p)} = 0\,. \tag{2.17}$$

Since neither the ghost two point function nor the mixing of the auxiliary field with the gauge field are renormalized in the axial gauge, it immediately follows that the two point function for the gauge field is not renormalized for this theory either. Similarly, one can show from eq. (2.16) (taking higher derivatives) that the gauge three point vertex, in this theory, is not renormalized as well, i.e. the theory is a *free theory*.

Thus, formally, we have shown that, in the axial gauge, the non-commutative pure Chern-Simons theory is a free theory and, consequently, the Chern-Simons coefficient is not renormalized at all. All of this, of course, depends heavily on the identities following from the vector supersymmetry invariance of the theory. Hence, such a conclusion will hold as long as we regularize the theory maintaining this symmetry. If, on the other hand, the regularization breaks this symmetry, such a conclusion will not hold and one may have anomalous contributions. It is worth pointing out here that, in literature, the pure Chern-Simons theory has been considered as the heavy mass limit of the Yang-Mills-Chern-Simons theory (namely, $m\to\infty$) [7, 16]. In this case, one can think of the Yang-Mills theory as providing a regularization. However, the Yang-Mills theory, itself, is not invariant under the vector supersymmetry transformations of eq. (2.15) (the same holds true in the Landau gauge as well.). Consequently, regularizing the theory in this manner violates the vector supersymmetry invariance and we would not expect our conclusion on the free nature of the theory to hold in this case. In fact, as is well known, the calculations from the Yang-Mills-Chern-Simons theory in the limit $m\to\infty$ show a shift of the Chern-Simons coefficient at the one-loop level. However, this shift is a quantized one in both non-commutative and commutative theories [8, 11, 17, 18]

3. One-loop calculations

The arguments of the last section on the free nature of the non-commutative pure Chern-Simons theory were based on algebraic identities and were quite formal. It is, therefore, essential to check these through an explicit one-loop calculation. This is also important from another point of view. Namely, in the axial gauge in commutative theories, one knows that a consistent prescription for handling unconventional poles is the principal value prescription. It is not clear whether such a prescription will continue to hold in the non-commutative case. Furthermore, an explicit calculation is likely to shade light on the question of ghost decoupling in closed loops in this theory.

The calculation of the self-energy, at one loop, is trivial. From the Feynman rules in eqs. (2.6), (2.8) and (2.9), one can write down the ghost and the gauge contribution to the gauge self-energy as (all momenta are defined as incoming and we are ignoring factors of $(2\pi)^3$ in the integrals)

$$\begin{aligned}
\Pi^{(1)\,AB(\text{ghost})}_{\mu\nu}(p) &= g^2 n^\mu n^\nu \int d^3k\, \frac{Q^{ACD}_\theta(p,-(k+p),k)Q^{BDC}_\theta(-p,-k,k+p)}{(n\cdot k)(n\cdot(k+p))}\,,\\
\Pi^{(1)\,AB(\text{gauge})}_{\mu\nu}(p) &= -g^2 n^\mu n^\nu \int d^3k\, \frac{Q^{ACD}_\theta(p,-(k+p),k)Q^{BDC}_\theta(-p,-k,k+p)}{(n\cdot k)(n\cdot(k+p))}\,.
\end{aligned} \tag{3.1}$$

The two contributions cancel each other and, consequently, the correction to the self-energy identically vanishes,

$$\Pi^{(1)\,AB}_{\mu\nu}(p) = \Pi^{(1)\,AB(\text{ghost})}_{\mu\nu}(p) + \Pi^{(1)\,(\text{gauge})}_{\mu\nu}(p) = 0\,. \tag{3.2}$$

This is completely consistent with our conclusions in the last section. Without giving details, we simply comment here that each of the two integrals in (3.1) can be evaluated (see discussion below for the triangle graph) and be shown to vanish individually. Thus, even if the ghost and the gauge loop corrections to self-energy did not cancel each other, their individual contributions would have been trivial. This is important because it shows that the ghosts do not contribute in the closed self-energy loop. On the other hand, the fact that the two contributions identically cancel each other has important consequence at finite temperature, as we will argue in the last section.

In analyzing the correction to the gauge interaction vertex, we note that, in the axial gauge, the usual Slavnov-Taylor identity, $Z_1 = Z_3$, would imply that there should be no correction to the triangle graph as well. However, if we naively write down the ghost and the gauge contributions to the triangle graph, they take the forms (One has to add the contribution from the crossed graph as well, which we are neglecting for the moment.)

$$\begin{aligned}
&\Gamma^{(1)\,ABC(\text{ghost})}_{\mu\nu\lambda}(p_1,p_2,-(p_1+p_2)) =\\
&\quad = ig^3 n_\mu n_\nu n_\lambda \int d^3k\, Q^{AA'C'}_\theta(p_1,-(k+p_1),k)Q^{BB'A'}_\theta(p_2,-(k+p_1+p_2),k+p_1)\times\\
&\qquad\qquad \times \frac{Q^{CC'B'}_\theta(-(p_1+p_2),-k,k+p_1+p_2)}{(n\cdot k)(n\cdot(k+p_1))(n\cdot(k+p_1+p_2))}\,,
\end{aligned}$$

$$
\begin{aligned}
&\Gamma^{(1)\,ABC(\text{gauge})}_{\mu\nu\lambda}(p_1,p_2,-(p_1+p_2)) = \\
&\quad = -2ig^3 n_\mu n_\nu n_\lambda \int d^3k\, Q_\theta^{AA'C'}(p_1,-(k+p_1),k) \times \\
&\qquad\qquad \times\, Q_\theta^{BB'A'}(p_2,-(k+p_1+p_2),k+p_1) \times \\
&\qquad\qquad \times\, \frac{Q_\theta^{CC'B'}(-(p_1+p_2),-k,k+p_1+p_2)}{(n\cdot k)(n\cdot(k+p_1))(n\cdot(k+p_1+p_2))}\,. \qquad (3.3)
\end{aligned}
$$

Therefore, the ghost and the gauge contributions do not seem to cancel each other in the triangle graph. Let us note that these graphs are logarithmic divergent by naive power counting and, therefore, if they do not vanish, then this would require adding counter terms that explicitly depend on n^μ. Such counter terms are known to exist in a general axial gauge [12], but do not arise in the homogeneous axial gauge (in commutative theories). Moreover, the triangle diagram, if it does not vanish would violate the Slavnov-Taylor identities of the theory. Therefore, we analyze the structure of these graphs carefully in some detail.

Let us recall that in the commutative gauge theories (in axial gauge) integrals of the kind

$$
\int d^n k\, \frac{1}{(n\cdot k)\cdots(n\cdot(k+p_1+\cdots+p_{m-1}))} \qquad (3.4)
$$

can all be regularized to zero using dimensional regularization [19]. (This is, in fact, one of the ways to see that ghosts decouple in the axial gauge.) It is not, however, clear whether this holds in non-commutative gauge theories, where the interaction vertices involve non-trivial phase factors. Let us, therefore, analyze the basic integral of the kind

$$
I = \int d^3k\, \frac{e^{ik\cdot p}}{(n\cdot k)(n\cdot(k+p_1))(n\cdot(k+p_1+p_2))}\,, \qquad (3.5)
$$

where p^μ is an arbitrary, external variable. First, we note that combining fractions, we can write the denominator of the integrand also as

$$
\begin{aligned}
&\frac{1}{(n\cdot p_1)(n\cdot(p_1+p_2))(n\cdot k)} - \frac{1}{(n\cdot p_1)(n\cdot p_2(n\cdot(k+p_1))} + \\
&\quad + \frac{1}{(n\cdot p_2)(n\cdot(p_1+p_2))(n\cdot(k+p_1+p_2))}\,. \qquad (3.6)
\end{aligned}
$$

Using eq. (3.6) in (3.5) and shifting variables of integration, we obtain

$$
I = \left[\frac{1}{(n\cdot p_1)(n\cdot(p_1+p_2))} - \frac{e^{-ip_1\cdot p}}{(n\cdot p_1)(n\cdot p_2)} + \frac{e^{-(p_1+p_2)\cdot p}}{(n\cdot p_2)(n\cdot(p_1+p_2))}\right]\bar{I}\,, \qquad (3.7)
$$

where

$$
\bar{I} = \int d^3k\, \frac{e^{ik\cdot p}}{(n\cdot k)} \qquad (3.8)
$$

which can be evaluated using the principal value prescription (standard in axial gauges) [20] to give

$$\bar{I}^{PV} = \delta^2(p^T)\int_{-\infty}^{\infty} d\omega\, \frac{e^{i\omega(n\cdot p)}}{\omega} = i\pi\,\mathrm{sgn}(n\cdot p)\delta^2(p^T)\,. \tag{3.9}$$

Here, we have decomposed p^μ into longitudinal and transverse directions with respect to n^μ and $p^{\mu T}$ represents the transverse component. We could also have used the Feynman prescription to evaluate the integral, which would have given

$$\bar{I}^{F} = -2i\pi\theta(-n\cdot p)\delta^2(p^T)\,, \tag{3.10}$$

where θ represents the conventional step function (and not the parameter of non-commutativity). There are several things to note here. First of all, unlike in the commutative theories (when there is no phase), such an integral is not automatically zero. Second, we note that, even in the presence of a phase factor, such an integral has a finite value whether we use the principal value prescription or the Feynman prescription for the pole, implying that there is no IR/UV mixing. In the following discussion, we will use the principal value prescription, but our conclusion, as will become clear, holds for the other case as well.

Using eq. (3.9) in (3.7), we can write

$$\begin{aligned}
&\int d^3k\, \frac{\cos k\cdot p}{(n\cdot k)(n\cdot(k+p_1))(n\cdot(k+p_1+p_2))} = \\
&\quad = -\pi\,\mathrm{sgn}(n\cdot p)\delta^2(p^T)\left[\frac{\sin p_1\cdot p}{(n\cdot p_1)(n\cdot p_2)} - \frac{\sin(p_1+p_2)\cdot p}{(n\cdot p_2)(n\cdot(p_1+p_2))}\right],\\
&\int d^3k\, \frac{\sin k\cdot p}{(n\cdot k)(n\cdot(k+p_1))(n\cdot(k+p_1+p_2))} = \\
&\quad = \pi\,\mathrm{sgn}(n\cdot p)\delta^2(p^T)\left[\frac{1}{(n\cdot p_1)(n\cdot(p_1+p_2))} - \frac{\cos p_1\cdot p}{(n\cdot p_1)(n\cdot p_2)} + \right.\\
&\qquad\qquad\left. + \frac{\cos(p_1+p_2)\cdot p}{(n\cdot p_2)(n\cdot(p_1+p_2))}\right].
\end{aligned} \tag{3.11}$$

These are the two basic integrals we will need to analyze the triangle graph. With these, let us go back to the triangle graph. Each of the integrands involves four possible group theoretic structures and using eq. (3.11) as well as simple trigonometric identities. It is easy to check that terms with an odd number of factors of d^{ABC} (namely, three d^{ABC} or two f^{ABC} and one d^{ABC}) identically vanish. On the other hand, terms with an even number of factors of d^{ABC} (no d^{ABC} or one f^{ABC} and two d^{ABC}) do not. It is worth noting that, as we expect, there are "planar" and "non-planar" parts, in the sense that after some trigonometric gymnastics, we find two generic types of integrals. One type has the θ dependent phase factor not involving the loop-momentum, these are the planar parts. Essentially the form of the planar loop integrals are the same as the commutative case and hence they vanish after regularization. The second type has the non-commutative phase factor dependent on loop-momentum k, and this is the non-planar part. These are the integrals we are concerned with here. The terms with three factors of f^{ABC} can be evaluated using eq. (3.11) and have the form (neglecting an overall factor)

$$I_1^{ABC} = \frac{\pi N}{4} f^{ABC} \sin\frac{p_1\theta p_2}{2}\left[\frac{\mathrm{sgn}(n\cdot\theta p_1)\delta^2((\theta p_1)^T)}{(n\cdot p_2)(n\cdot(p_1+p_2))} + \frac{\mathrm{sgn}(n\cdot\theta p_2)}{(n\cdot p_1)(n\cdot(p_1+p_2))} + \right.$$
$$\left. + \frac{\mathrm{sgn}(n\cdot\theta(p_1+p_2)\delta^2((\theta(p_1+p_2))^T)}{(n\cdot p_1)(n\cdot p_2)}\right]. \tag{3.12}$$

The terms involving one factor of f and two d's can similarly be evaluated to give

$$I_2^{ABC} = -\frac{\pi N}{4} f^{ABC} \sin\frac{p_1\theta p_2}{2}\left[\frac{\mathrm{sgn}(n\cdot\theta p_1)\delta^2((\theta p_1)^T)}{(n\cdot p_2)(n\cdot(p_1+p_2))} + \frac{\mathrm{sgn}(n\cdot\theta p_2)}{(n\cdot p_1)(n\cdot(p_1+p_2))} + \right.$$
$$\left. + \frac{\mathrm{sgn}(n\cdot\theta(p_1+p_2)\delta^2((\theta(p_1+p_2))^T)}{(n\cdot p_1)(n\cdot p_2)}\right]. \tag{3.13}$$

Here, we have used various group theoretic identities (given in [15]) to simplify the final structure. It is clear now, that even though the individual contributions to the triangle graph of the structures fff and fdd are finite and non zero, the sum identically vanishes.

$$I_1^{ABC} + I_2^{ABC} = 0\,. \tag{3.14}$$

In other words, each of the two integrands in eqs. (3.3) individually vanishes and, consequently, there is no correction to the gauge interaction vertex. This is, of course, exactly what we had concluded from the identities following from vector supersymmetry. More importantly, though, it shows that the ghosts in closed loops decouple in this theory, both in the self-energy and the triangle graphs. With these results, one can show from eq. (2.12), with a little bit of analysis, that in this theory, for $n \geq 4$,

$$p_{n,\mu_n}\Gamma^{\mu_1,\ldots,\mu_n}_{A_1,\ldots,A_n}(p_1,\ldots,p_n) = 0 \tag{3.15}$$

which is true for every external momentum. The n-point function, of course, gets contributions from a ghost loop as well as a gauge loop. Both of them have the same structure (as we have already seen in the case of the triangle graph), but different coefficients (the different coefficient for the gauge loop arises primarily because of the contractions of the $\epsilon^{\mu\nu\lambda}$ tensors). Therefore, they cannot cancel each other. Rather, the ghost loop as well as the gauge loop have to be individually transverse to the external momenta. However, each of these graphs is proportional to $n^{\mu_1,\ldots,\mu_n}$ and, therefore, can be transverse to the external momenta provided it is proportional to $\delta(n\cdot p_1)\cdots\delta(n\cdot p_n)$. On the other hand, by the method of combining fractions, we can show that each of these graphs is proportional to $\bar{I}$ (multiplied by factors of trigonometric functions of external momenta) which we have already evaluated and which does not have this property. Consequently, it follows that, for eq. (3.15) to hold, each of the graphs must vanish independently. This, therefore, shows that the theory is truly free. More than that it shows that ghosts in closed loops do not contribute. With our earlier result that ghosts in open lines do not contribute in the axial gauge, it is clear that, in this theory, ghosts completely decouple. Furthermore, since the structure of the ghost propagator and the ghost interaction vertex in the axial gauge is independent of the nature of the gauge theory we are considering (we may have a more complicated theory such as a non-commutative Yang-Mills-Chern-Simons theory or a non-commutative Yang-Mills theory, for example), the ghost loop will vanish independent of the theory we are considering. Therefore, we have proved the more general result that the ghosts completely decouple in the axial gauge in a non-commutative gauge theory.

4. Conclusion

In this paper, we have studied systematically the structure of non-commutative pure Chern-Simons theory in the homogeneous axial gauge. There are many interesting results that one finds in this theory. First, of all, we have shown that there is no problem in using the homogeneous axial gauge in such theories and the principal value prescription makes individual integrals well behaved. At one loop, we have shown that there is no quantum correction, which is consistent with the conclusion that this is a free theory. Such a conclusion follows from the identities coming from the vector supersymmetry of the theory together with the usual Slavnov-Taylor identities. Our one-loop calculation shows that there is no IR/UV mixing and that the Slavnov-Taylor identities hold in such theories in the axial gauge. In fact, the quantum behavior of this non-commutative theory is very much like its commutative counterpart [11]. Without going into details, let us simply note here that, since Slavnov-Taylor identities hold in the axial gauge in this theory, by a generalization of the arguments in [11], it can also be shown, in the non-commutative Yang-Mills-Chern-Simons theory, that there is no correction to the Chern-Simons coefficient beyond one loop in the axial gauge. Another way of saying this is that the ratio $4\pi m/g^2$ receives no correction beyond one loop in any gauge. This is quite important from the point of view of the consistency of the theory under *large gauge* transformations. Finally, let us note that it has been proposed to use the non-commutative pure Chern-Simons theory to describe quantum Hall effect. In this connection, the behavior of the Chern-Simons coefficient at finite temperature is quite crucial, if it were to describe a phase transition. We note that, at finite temperature, gauge fields and ghosts obey the *same statistics* [21] so that the finite temperature propagators, in the axial gauge (in real time formalism), have the forms

$$\begin{aligned}
D_{\mu\nu}^{AB\,(\beta)}(p) &= \frac{i\delta^{AB}}{m}\epsilon_{\mu\nu\lambda}n^{\lambda}\left(\frac{1}{n\cdot p} - 2i\pi n_B(|p^0|)\delta(n\cdot p)\right), \\
D^{AB\,(\beta)}(p) &= i\delta^{AB}\left(\frac{1}{n\cdot p} - 2i\pi n_B(|p^0|)\delta(n\cdot p)\right).
\end{aligned} \tag{4.1}$$

Since there is no modification of the interaction vertices at finite temperature, from the forms of eqs. (3.1) and (4.1), it is clear that the correction to the self-energy will vanish even at finite temperature in the non-commutative pure Chern-Simons theory. As a result, the Chern-Simons coefficient cannot have a temperature dependence and, consequently, it would appear that such a model may not have a satisfactory behavior to describe quantum Hall phase transitions. More precisely, the quantum Hall fluid-Wigner crystal phase transition cannot be driven by temperature effects. So, it seems that the non-commutative Chern-Simons approximation is not enough for describing this phase transition.

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